Syntactic approximations to computational complexity classes

Argimiro Arratia[1]*** and Carlos E. Ortiz[2]†

[1] Dpto. de Matemática Aplicada, Facultad de Ciencias, Universidad de Valladolid, Valladolid 47005, Spain, arratia@mac.cie.uva.es,

[2] Department of Mathematics and Computer Science, Arcadia University, 450 S. Easton Road, Glenside, PA 19038-3295, U.S.A. ortiz@arcadia.edu

Abstract. We present a formal syntax of approximate formulas suited for the logic with counting quantifiers $\mathcal{SOLP}$. This logic was studied by us in [1] where, among other properties, we showed: (i) In the presence of a built–in (linear) order, $\mathcal{SOLP}$ can describe **NP**–complete problems and fragments of it capture classes like **P** and **NL**; (ii) weakening the ordering relation to an almost order we can separate meaningful fragments, using a combinatorial tool adapted to these languages.

The purpose of the approximate formulas presented here, is to provide a syntactic approximation to logics contained in $\mathcal{SOLP}$ with built-in order, that should be complementary of the semantic approximation based on almost orders, by producing approximating logics where problems are described within a small counting error. We state and prove a Bridge Theorem that links expressibility in fragments of $\mathcal{SOLP}$ over almost-ordered structures to expressibility with respect to approximate formulas for the corresponding fragments over ordered structures. A consequence of these results is that proving inexpressibility results over fragments of $\mathcal{SOLP}$ with built-in order could be done by proving inexpressibility over the corresponding fragments with built-in almost order, where separation proofs are allegedly easier.

Subject Classification: Logic in computer science; Descriptive Complexity.

1 Introduction

Descriptive Complexity deals mainly with producing logics that define all problems of particular computational complexity, and adapting the classical tools for showing inexpressibility of queries in logics in the context of finite models, in the hope to obtain worthy lower bounds for computational classes such as **P** or **NP**. The limitations of this logical approach to showing computational complexity bounds for classes like say, **P**, **NL** (*nondeterministic logspace*), and

*** Supported by grants MOISES (TIN2005-08832-C03-02) and SINGACOM (MTM2004-00958), MEC–Spain

† Supported by a Faculty Award Grant from the Christian R. & Mary F. Lindback Foundation, and a Visiting Research Fellowship from Universidad de Valladolid

others within **NP**, boils down to the fact that, as of today, all known logics that define problems in these classes need a relation of linear order built–in into their semantics. In the presence of this built–in linear order, logical inexpressibility tools such as Ehrenfeucht-Fraïssé games have little power for telling structures apart (e.g. see [4, § 6.6]). But, on the other hand, in the absence of this built–in linear order, logics loose significantly expressive power: For example, first order logic (FO) extended with a least fixed point operator (LFP(FO)) with order captures all of **P** (in the sense that it is capable of defining all polynomial time computable properties), but without order can not express the parity of the size of a set. To overcome this difficulty, a natural idea is to study *approximations* to logics with built–in order, where techniques like Ehrenfeucht-Fraïssé games become effective in showing separability results, and hopefully these separations in the approximate setting will give a clue on how to go about separating the associated logics with order.

There are two main approaches to define approximate logics in model theory. One is to play with the semantics, where constructs as built–in order is weakened to an "*almost–order*", and, frequently, some counting operator is added to compensate for the loss of expressive power. This has been the typical approach within the Descriptive Complexity community (e.g. [2], [6] among others), and it has some severe limitations: For example, the paper by Libkin and Wong [6] shows that a very powerful extension of first order logic with additional counting quantifiers, known as $\mathcal{L}^*_{\infty\omega}(C)$, which subsumes various counting extensions of FO, in the presence of almost–orders has the *bounded number of degrees property* (or BNDP) and thus cannot express the transitive closure of a binary relation.

The other approach is syntactic and is found in classical model theory as in, for example, Keisler's logic of probability quantifiers (see [5]), who conceived it as a logic appropriate for his investigations on *probability hyperfinite spaces*, or infinite structures suitable for approximating large finite phenomena of applied mathematics. Under this approach, for each formula φ of a logic and every real number ϵ one construct an approximate formula φ_ϵ with the property that in every model $\mathcal{A}$, if $\epsilon_1 < 0 < \epsilon_2$ then $\varphi_{\epsilon_1} \to \varphi \to \varphi_{\epsilon_2}$, and as ϵ tends to 0, the interpretation of φ_ϵ should be closer to φ. This approach has been developed with success in the theory of classical metric spaces but not, to our knowledge, in Computational Complexity theory.

In this paper we develop a syntactic approach to the task of approximating logics with built–in order based on a notion of approximate formulas *á la* Keisler, and show how it relates to the semantic approach based on almost orders. This approach is potentially relevant to the problem of separating logics with built-in order, since we obtain a result that implies that separation of logics with built-in almost-order can be translated into separation of corresponding logics with built-in order.

The framework for our results is the second order logic of proportionality quantifiers, $\mathcal{SOLP}$, defined in [1]. The quantifiers for this logic are counting quantifiers acting upon second order terms. When restricted to built-in almost orders this logic avoids the BNDP, has non trivial expressive power, and general separation results of combinatorial nature can be obtained. We review the definition of $\mathcal{SOLP}$ and summarise facts found in [1] about its expressive power in the presence of almost orders in section 2. In section 3 we introduce the new syntax of approximate formulas suited for $\mathcal{SOLP}$, and prove a Bridge Theorem which establish a correspondence between satisfaction of formulas in $\mathcal{SOLP}$ in almost ordered structures and satisfaction of the corresponding approximate formulas in ordered structures. In section 4 we introduce the notion of the ϵ–approximate logic $\mathcal{L}_\epsilon$, for every fragment $\mathcal{L}$ of $\mathcal{SOLP}$; a logic that should have an expressive power "almost" similar to the expressive power of $\mathcal{L}$. This notion in turn generates the notions of strong expressibility and ϵ–relaxed fragments. An ϵ–relaxed fragment is one for which $\mathcal{L}_\delta = \mathcal{L}$ (in terms of expressive power) for every $\delta \in (-\epsilon, \epsilon)$. Surprisingly, fragments of $\mathcal{SOLP}$ with built–in order that capture **P** and **NL** are ϵ–relaxed. A nice property of ϵ–relaxed logics is that for them strong expressibility and expressibility are "almost" equivalent (an idea that we will formalise). A consequence of this is Theorem 4 that shows that to prove inexpressibility of problems in ϵ–relaxed logics *with built–in order* it is enough to prove inexpressibility of the same problem in the δ-approximate logics ($\delta \in (-\epsilon, \epsilon)$) with respect to *almost ordered* structures. Since proving inexpressibility for logics over almost orders is easier, in practice, than the usual checking of satisfaction in ordered structures, this last result has potential applicability for studying separation of well known logics with built-in order, such as the ones that capture **NL** and **P**.

2 The second order logic of proportional quantifiers

Definition 1. *The Second Order Logic of Proportional quantifiers, $\mathcal{SOLP}$, is the set of formulas of the form*

$$Q_1 \cdots Q_u \theta(x_1, \ldots, x_s, X_1, \ldots, X_r) \tag{1}$$

where $\theta(x_1, \ldots, x_s, X_1, \ldots, X_r)$ is a first order formula over some vocabulary τ with (free) first order variables $x_1, \ldots, x_k$ and second order variables, $X_1, \ldots, X_r$; each Q_j $(j \leq u)$ is either $(P(X_i) \geq t_i)$ or $(P(X_i) \leq t_i)$, where t_i is a rational in $(0,1)$, for some $i \leq r$. Whenever we want to make the underlying vocabulary τ explicit we will write $\mathcal{SOLP}(\tau)$.

We also define $\mathcal{SOLP}(\tau)[r_1, \ldots, r_k]$, for a given vocabulary τ and sequence $r_1, r_2, \ldots, r_k$ of distinct natural numbers, as the sublogic of $\mathcal{SOLP}(\tau)$ where the proportional quantifiers can only be of the form $(P(X) \leq q/r_i)$ or $(P(X) \geq q/r_i)$, for $i = 1, \ldots, k$ and q a natural number such that $0 < q < r_i$.

Another fragment of $\mathcal{SOLP}$ which will be of interest for us is the Second Order Monadic Logic of Proportional quantifiers*, denoted $\mathcal{SOMLP}$, which is $\mathcal{SOLP}$ with the arity of the second order variables in* (1) *being all equal to 1.*

The interpretation for the proportional quantifiers is very natural: Let X be a second order variable of arity k, $\overline{Y}$ a vector of second order variables, $\overline{x} = x_1, \ldots, x_m$ first order variables and $\phi(\overline{x}, \overline{Y}, X)$ a formula in $\mathcal{SOLP}(\tau)$ over some (finite) vocabulary τ (which does not contains X or any of the variables in $\overline{Y}$ as a relation symbol). Let r be a rational in $(0,1)$. Then

$$(P(X) \geq r)\phi(\overline{x}, \overline{Y}, X) \quad \text{and} \quad (P(X) \leq r)\phi(\overline{x}, \overline{Y}, X)$$

have the following semantics. For an appropriate finite τ–structure $\mathcal{A}$, elements $\overline{a} = (a_1, \ldots, a_m)$ in A and an appropriate vector of relations $\overline{B}$ over A, we have

$\mathcal{A} \models (P(X) \geq r)\phi(\overline{a}, \overline{B}, X) \iff$ there exists $S \subseteq A^k$ such that $\mathcal{A} \models \phi(\overline{a}, \overline{B}, S)$ and $|S| \geq r \cdot |A|^k$

Similarly for $(P(X) \leq r)\phi(\overline{x}, \overline{Y}, X)$, substituting $\geq$ for $\leq$ above.

Example 1. Let $\tau = \{R, s, t\}$ where R is a ternary relation symbol, and s and t are constant symbols. Let r be a rational with $0 < r < 1$. We define

NOT-IN-CLOS$_{\leq r}$ **:=** $\{\mathcal{A} = \langle A, R, s, t\rangle :$ A has a set containing s but not t, closed under R, and of size at most a fraction r of $|A|$ $\}$.

Let $\beta_{nclos}(X)$ be the following formula

$$\begin{aligned}\beta_{nclos}(X) := \forall x \forall u \forall v\ [X(s) \wedge \neg X(t) \\ \wedge\ (X(u) \wedge X(v) \wedge R(u, v, x) \rightarrow X(x))]\end{aligned}$$

Then $\mathcal{A} \in$ NOT-IN-CLOS$_{\leq r} \iff \mathcal{A} \models (P(X) \leq r)\beta_{nclos}(X)$.
In [1] it is shown that, for $r = 1/n$, this problem is complete for **P** under first order reductions.

Example 2. Let $\tau = \{E, s\}$ where E is a binary relation symbol and s is a constant symbol. We think of τ-structures as graphs or digraphs (directed graphs) with a specified vertex s (the source). Let r be a rational with $0 < r < 1$. We define

NCON$_{\geq r}$ **:=** $\{\mathcal{A} = \langle A, E, s\rangle : \langle A, E\rangle$ is a digraph and at least a fraction r of the vertices are **not** connected to $s\}$

Let $\alpha_{ncon}(Y)$ be the following formula

$$\alpha_{ncon}(Y) := \neg Y(s) \wedge \forall x \forall y (E(x, y) \wedge Y(x) \rightarrow Y(y))$$

Then $\mathcal{A} \in$ NCON$_{\geq r} \iff \mathcal{A} \models (P(Y) \geq r)\alpha_{ncon}(Y)$.
We proved NCON$_{\geq 1/2}$ is complete for **NL** under first order reductions (see [1]).

2.1 Summary of facts about $\mathcal{SOLP}$

In [1] we study the expressive power of $\mathcal{SOLP}$ in the presence of built–in order and when this external predicate is weakened to an almost order (see [4] for the notion and use of built–in numerical predicates in Descriptive Complexity). We summarise below the facts from [1] that we need about what could be called "semantic approximations" to definability in $\mathcal{SOLP}$. We have shown that:

(1) In the presence of order (at least a built–in successor), $\mathbf{P} \subseteq \mathcal{SOLP}[2]$ and, furthermore, it is captured by the fragment $\mathcal{SOLP}Horn[2]$, consisting of formulas of the form $(P(X_1) \leq 1/2) \cdots (P(X_r) \leq 1/2)\alpha$, where α is a universal Horn formula over some vocabulary τ and second order variables $X_1, \ldots, X_r$.
(2) In the presence of order, $\mathbf{NL}$ is captured by $\mathcal{SOLP}Krom[2]$, a fragment consisting of formulas of the form $(P(X_1) \geq 1/2) \cdots (P(X_r) \geq 1/2)\alpha$, where α is a universal Krom formula. (This and the previous capturing of $\mathbf{P}$ by fragments of $\mathcal{SOLP}$ are inspired on Grädel's [3], but taking into account the limitations in the cardinalities of second order variables imposed by our counting quantifiers.)
(3) With respect to almost ordered structures we have an infinite hierarchy within the monadic fragment $\mathcal{SOMLP}$, namely,

$$\mathcal{SOMLP}[2] \subsetneq \mathcal{SOMLP}[2,3] \subsetneq \mathcal{SOMLP}[2,3,5] \subsetneq \ldots$$

(4) With respect to almost ordered structures and unbounded arity we have that

$$\mathcal{SOLP}Horn[2] \subsetneq \mathcal{SOLP}[2,3].$$

The separation results listed in (3) and (4) were obtained with appropriate Ehrenfeucht–Fraïssé games. The concept of almost order (taken from [6]) constitutes the core of our "semantic approximations", around which we work our syntactic approximations, and thus we pause to review this concept and further constructions from [1].

Definition 2. *A function $g : \mathbb{N} \to \mathbb{N}$ is sublinear if, for all $n \in \mathbb{N}$, $g(n) < n$. For a fixed positive integer k, a k-preorder over a set A is a binary, reflexive and transitive relation P in which every induced equivalence class of $P \cap P^{-1}$ has size at most k. An almost linear order over a set A of cardinality n, determined by a sublinear function $g : \mathbb{N} \to \mathbb{N}$, is a binary relation $\leq_g$ over A with a partition of the universe A into two sets B, C, such that B has cardinality $n - g(n)$ and $\leq_g$ restricted to B is a linear order, $\leq_g$ restricted to C is a 2-preorder, and for every $x \in C$ and every $y \in B$, $x \leq_g y$, but $y \not\leq_g x$.*

Note that for any function $g : \mathbb{N} \to \mathbb{N}$, the almost linear order $\leq_g$ over a set A induces an equivalence relation $\sim_g$ in A defined by $a \sim_g b$ iff $a \leq_g b$ and $b \leq_g a$. For $a \in A$, let $[a]_g$ denote its $\sim_g$–equivalence class, and $[A]_g := \{[a]_g : a \in A\}$.

Definition 3. *Fix a sublinear $g : \mathbb{N} \to \mathbb{N}$ and let R be a k-ary relation on a set A. Let $\leq_g$ be an almost order determined by g in A. We say that R is consistent with $\leq_g$ if for every pair of vectors $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ of elements in A with $a_i \sim_g b_i$ for every $i \leq k$, we have that*
$R(a_1, \ldots, a_k)$ holds if and only if $R(b_1, \ldots, b_k)$ holds.
Let $\mathcal{A} = \langle A, R_1^{\mathcal{A}}, \ldots, R_t^{\mathcal{A}}, C_1^{\mathcal{A}}, \ldots, C_s^{\mathcal{A}} \rangle$ be a τ-structure. We say that $\mathcal{A}$ is consistent with $\leq_g$ if and only if for every $i \leq t$, $R_i^{\mathcal{A}}$ is consistent with $\leq_g$.

For a τ-structure $\mathcal{A}$, consistent with $\leq_g$, it makes sense to define the *quotient structure* $\mathcal{A}/_{\sim_g}$, as a τ-structure consisting of $[A]_g$ as its universe, and for a k-ary relation $R \in \tau$,

$$R^{\mathcal{A}/\sim_g} := \{([a_1]_g, \ldots, [a_k]_g) : (a_1, \ldots, a_k) \in R^{\mathcal{A}}\}$$

Furthermore, for $B \subseteq A$ we define its $\leq_g$*-contraction* as $[B]_g := \{[b]_g : b \in B\}$.

By $\mathcal{SOLP}+\leq_g$, for an almost order $\leq_g$, we understand the logic $\mathcal{SOLP}$ with the almost order $\leq_g$ as additional built-in relation, and where we only consider models $\mathcal{A}$ that are consistent with $\leq_g$. Furthermore, for the formulas of the form $(P(X) \geq r)\phi(\overline{x}, \overline{Y}, X)$ and $(P(X) \leq r)\phi(\overline{x}, \overline{Y}, X)$, we require the following modification of the semantics: For an appropriate finite model $\mathcal{A}$ consistent with $\leq_g$, for elements $\overline{a} = (a_1, \ldots, a_m)$ in A and an appropriate vector of relations $\overline{B}$, consistent with $\leq_g$, we should have

$$\mathcal{A} \models (P(X) \geq r)\phi(\overline{a}, \overline{B}, X) \iff \begin{array}{l} \text{there exists } S \subseteq A^k, \text{ consistent with } \leq_g, \\ \text{such that } \mathcal{A} \models \phi(\overline{a}, \overline{B}, S) \text{ and } |S| \geq r \cdot |A|^k \end{array}$$

Similarly for $(P(X) \leq r)\phi(\overline{x}, \overline{Y}, X)$, substituting $\geq$ for $\leq$ above.
The property of being consistent for $\leq_g$ extends, by induction, to all the formulas in $\mathcal{SOLP}(\tau)_{\leq_g}$.

Remark 1. In general, given a logic $\mathcal{L} \subseteq \mathcal{SOLP}$, we use $\mathcal{L}+ \leq_g$ to indicate that all possible (finite) models of $\mathcal{L}$ have an almost order $\leq_g$, determined by a sublinear function g. Also $\mathcal{L}+ \leq$ indicates that the models have an additional linear order.

3 A syntax of approximate formulas

We now introduce the notion of approximate formulas for $\mathcal{SOLP}$. The purpose of these formulas is to provide a link between satisfaction in almost ordered structures and satisfaction in their corresponding quotient structures. This we will make precise in the Bridge Theorem (Theorem 1 below); thus, whatever we can say about a class of almost ordered structures we can "approximately" say about a class of their quotient structures, and vice versa.

Definition 4 (Approximate Formulas). *For every $\epsilon \in [0, 1)$ and for every formula $\theta(\overline{x}, \overline{X}) \in \mathcal{SOLP}(\tau)$, we define the positive (resp. negative) ϵ-approximation of $\theta(\overline{x}, \overline{X})$, denoted $\theta(\overline{x}, \overline{X})_\epsilon$ (resp. $\theta(\overline{x}, \overline{X})_{-\epsilon}$), as follows:*

First order formulas *If $\theta(\overline{x}, \overline{X})$ is a first order formula with free second order variables among the $\overline{X}$ and free first order variables among the $\overline{x}$, then $\theta(\overline{x}, \overline{X})_\epsilon = \theta(\overline{x}, \overline{X})_{-\epsilon} := \theta(\overline{x}, \overline{X})$.*

Proportional quantifiers *If $\theta(\overline{x}, \overline{X}) := (Q_1 \ldots Q_u)\varphi(\overline{x}, \overline{X})$, where $\varphi(\overline{x}, \overline{X})$ is a first–order formula and $Q_1, \ldots, Q_u$ are proportional quantifiers, its ϵ-approximation is the $\mathcal{SOLP}$-formula $(\theta(\overline{x}, \overline{X}))_\epsilon := (Q'_1 \ldots Q'_u)\varphi(\overline{x}, \overline{X})$, where, for each j, the proportional quantifier Q'_j is chosen as follows:*
(a) If Q_j is of the form $(P(Y) \geq r)$, where Y is of arity $k \geq 1$, then Q'_j is of the form

$$\begin{cases} (P(Y) \geq (1-\epsilon)^{k-1}[r - k\epsilon]) & \textit{if } \ r - k\epsilon > 0 \\ (P(Y) \geq 0) & \textit{otherwise} \end{cases}$$

(b) If Q_j is of the form $(P(Y) \leq r)$, then Q'_j is of the form

$$\begin{cases} (P(Y) \leq (1+\epsilon)^{k-1}[r+k\epsilon]) & \text{if } (1+\epsilon)^{k-1}(r+k\epsilon) < 1 \\ (P(Y) \leq 1) & \text{otherwise} \end{cases}$$

And the negative approximation, $\theta(\overline{x}, \overline{X})_{-\epsilon} := (Q'_1 \dots Q'_u)\varphi(\overline{x}, \overline{X})$, is given by:
(a) If Q_j is of the form $(P(Y) \geq r)$, where Y is of arity $k \geq 1$, then Q'_j is of the form

$$\begin{cases} (P(Y) \geq \frac{1}{(1-\epsilon)^{k-1}}[r+k\epsilon(1-\epsilon)^{k-1}]) & \text{if } \frac{r}{(1-\epsilon)^{k-1}} + k\epsilon < 1 \\ (P(Y) \geq 1) & \text{otherwise} \end{cases}$$

(b) If Q_j is of the form $(P(Y) \leq r)$, then Q'_j is of the form

$$\begin{cases} (P(Y) \leq \frac{1}{(1+\epsilon)^{k-1}}[r-k\epsilon(1-\epsilon)^{k-1}]) & \text{if } \frac{r}{(1+\epsilon)^{k-1}} - k\epsilon > 0 \\ (P(Y) \leq 0) & \text{otherwise} \end{cases}$$

Remark 2. We can always assume that ϵ is small enough so that the ϵ–approximation for formulas with proportional quantifiers is the first option in their definition, e.g., for $(P(Y) \leq r)\varphi(Y)$ it will always be $(P(Y) \leq (1+\epsilon)^{k-1}[r+k\epsilon])\varphi(Y)_\epsilon$.

Remark 3. Observe that when Y is monadic (has arity 1) the ϵ–approximation of a formula preceded by quantifier $(P(Y) \geq r)$ (resp. $(P(Y) \leq r)$) is the ϵ–approximation of the formula preceded by quantifier $(P(Y) \geq r - \epsilon)$ (resp. $(P(Y) \leq r + \epsilon)$), which is what one would expect in this case. Our definition for a Y of any arity may seem awkward, but it is the right one for establishing a correspondence between satisfaction of formulas in $\mathcal{SOLP}$ in almost ordered structures and satisfaction of the corresponding approximate formulas in ordered structures, as we shall prove below.

The basic link between positive and negative approximate formulas, and the formula that they approximate is given by the following lemma.

Lemma 1. *For every formula $\theta(\overline{x}, \overline{X}) \in \mathcal{SOLP}(\tau)$, for every finite τ–structure $\mathcal{A}$, for every interpretation $\overline{A}$ of relation symbols $\overline{X}$ in $\mathcal{A}$, for every tuple of elements $\overline{a}$ in $\mathcal{A}$ and for ϵ and δ such that $0 < \delta < \epsilon < 1$, we have that:*

$$\mathcal{A} \models \theta(\overline{a}, \overline{A})_{-\epsilon} \rightarrow \theta(\overline{a}, \overline{A})_{-\delta} \rightarrow \theta(\overline{a}, \overline{A}) \rightarrow \theta(\overline{a}, \overline{A})_{\delta} \rightarrow \theta(\overline{a}, \overline{A})_{\epsilon}.$$

Furthermore, for every formula $\theta(\overline{x}, \overline{X}) \in \mathcal{SOLP}(\tau)$, for every ϵ with $0 < \epsilon < 1$

$$(\theta(\overline{x}, \overline{X})_{-\epsilon})_\epsilon = \theta(\overline{x}, \overline{X}) = (\theta(\overline{x}, \overline{X})_\epsilon)_{-\epsilon}. \qquad \square$$

We will now show that it is possible to jump from satisfaction in almost order (respectively, linearly ordered) structures to satisfaction of approximate formulas in linearly ordered (respectively, almost ordered) structures.

Theorem 1 (Bridge Theorem). *Fix a sublinear function g and an almost order $\leq_g$. For every formula $\theta(x_1, \ldots, x_k, X) \in \mathcal{SOLP}(\tau)$, for every τ-structure $\mathcal{A}$ of size m and consistent with $\leq_g$, for every $\overline{a} = (a_1, \ldots, a_k) \in A^k$, for every predicate S of arity $t \geq 1$, the following holds:*

(i) $\mathcal{A} \models \theta(\overline{a}, S)$ *implies* $\mathcal{A}/_{\sim_g} \models \theta([\overline{a}]_g, [S]_g)_{\gamma(m)}$, *where* $\gamma(m) = \dfrac{g(m)}{2m - g(m)}$

(ii) $\mathcal{A}/_{\sim_g} \models \theta([\overline{a}]_g, [S]_g)$ *implies* $\mathcal{A} \models \theta(\overline{a}, S)_{\beta(m)}$, *where* $\beta(m) = \dfrac{g(m)}{2m}$

(iii) $\mathcal{A} \models \theta(\overline{a}, S)_{-\gamma(m)}$ *implies* $\mathcal{A}/_{\sim_g} \models \theta([\overline{a}]_g, [S]_g)$, *$\gamma$ as in (i)*

(iv) $\mathcal{A}/_{\sim_g} \models \theta([\overline{a}]_g, [S]_g)_{-\beta(m)}$ *implies* $\mathcal{A} \models \theta(\overline{a}, S)$, *$\beta$ as in (ii)*

Proof. (see the Appendix.) □

The picture that we have relating satisfaction in the almost ordered world with satisfaction in the ordered world is the following (the horizontal arrows are given by lemma 1 and the diagonal arrows by the Bridge Theorem):

$$\begin{tikzcd}
\mathcal{A} \models & \theta_{-\gamma} \arrow[r] \arrow[dr] & \theta \arrow[r] \arrow[dr] & \theta_{\beta} & \text{(almost order)} \\
\mathcal{A}/_{\sim_g} \models & \theta_{-\beta} \arrow[r] \arrow[ur] & \theta \arrow[r] \arrow[ur] & \theta_{\gamma} & \text{(order)}
\end{tikzcd}$$

Now the ground is set. From experience we know that inexpressibility results are, in general, easier to accomplish in the presence of almost order, but to transfer these separations to the truly (linearly) ordered world is hard. Our picture shows that, in fact, the passing from the almost ordered world to a corresponding ordered world (or vice versa) changes the syntactic description of some problem for an approximate description. Is an approximate description as good as an exact description for determining inexpressibility of a class of ordered structures? We feel that the answer to this last question is "yes in almost all cases", and in the remainder of this paper we give formal support to this intuition.

4 Strong expressibility

We define the idea of strong equivalence for two formulas as follows.

Definition 5. *Fix two sentences $\phi, \psi \in \mathcal{SOLP}$. We say that ϕ* **is strongly equivalent** *to ψ (in symbols $\phi \Leftrightarrow_S \psi$) iff there exists $\epsilon \in (0, 1)$ such that in every model $\mathcal{A}$: $\mathcal{A} \models \phi_\epsilon \rightarrow \psi_{-\epsilon}$ and $\mathcal{A} \models \psi_\epsilon \rightarrow \phi_{-\epsilon}$.*

The intuition is that two sentences that are strongly equivalent can be syntactically approximate as much as we like. Formally what this means is that, if $\phi \Leftrightarrow_S \psi$ then there exists an $\epsilon > 0$ such that for every $\beta, \gamma \in (-\epsilon, \epsilon)$, $\models \phi \leftrightarrow \phi_\beta \leftrightarrow \psi_\gamma \leftrightarrow \psi$.

Note that if ϕ is strongly equivalent to ψ then for every model $\mathcal{A}$, $\mathcal{A} \models \phi \leftrightarrow \psi$ (i.e. ϕ and ψ are equivalent). Note also that it is not clear at all that $\phi \Leftrightarrow_S \phi$. The next example proves that this happens sometimes.

Example 3. The property of being 2-colorable can be expressed in $\mathcal{SOMLP}[2]$ as follows. Let X and Y be two unary second order variables, and let $\theta(X, Y)$ be a formula that says that

$$
\begin{gathered}
(X \text{ and } Y \text{ are disjoint }) \;\wedge \\
\forall x \forall y(((X(x) \vee Y(x)) \wedge E(x,y) \rightarrow \neg(X(y) \vee Y(y))) \;\wedge \\
(\neg(X(x) \vee Y(x)) \wedge E(x,y) \rightarrow (X(y) \vee Y(y)))
\end{gathered}
$$

Then the $\mathcal{SOMLP}[2](\{E\})$ sentence

$$\phi := (P(X) \leq (1/2))(P(Y) \leq (1/2))\theta(X, Y)$$

holds in a graph, if and only if the graph is 2-colorable. (The idea is that X and Y constitute a partition of one of the possible two colors; in fact the color applied to the fewest number of vertices.)

Now observe that for every $\epsilon << 1/2$, if $\mathcal{A} \models \phi_\epsilon$ then

$$\mathcal{A} \models (P(X) \leq (1/2+\epsilon))(P(Y) \leq (1/2+\epsilon))\theta(X, Y)$$

It follows that $\mathcal{A}$ is 2-colorable, and in consequence

$$\mathcal{A} \models \phi_{-\epsilon} := (P(X) \leq (1/2-\epsilon))(P(Y) \leq (1/2-\epsilon))\theta(X, Y)$$

We proceed to define the approximate logics.

Definition 6. *Fix a logic $\mathcal{L} \subseteq \mathcal{SOLP}$ and an $\epsilon \in (-1, 1)$. The ϵ-approximation of $\mathcal{L}$, denoted $\mathcal{L}_\epsilon$, is the following fragment of $\mathcal{SOLP}$: $\{\phi_\epsilon : \phi \in \mathcal{L}\}$.*

By convention we define $\mathcal{L}_0 = \mathcal{L}$. The approximation of $\mathcal{L}$ (or the approximate logic corresponding to $\mathcal{L}$) is the set of formulas $\mathcal{L}_A := \bigcup_{\epsilon \in (-1,1)} \mathcal{L}_\epsilon$

We are interested in fragments of $\mathcal{SOLP}$ that behave "decently" for the notion of strong equivalence, i.e. where at least we can ask that for every formula ϕ in the fragment, $\phi \Leftrightarrow_S \phi$.

Definition 7. *We say that a fragment $\mathcal{L}$ of $\mathcal{SOLP}$ is* **ϵ-relaxed** *if, for every $\delta \in (-\epsilon, \epsilon)$, $\mathcal{L}_\delta = \mathcal{L}$ (i.e., their expressive power is the same).*

Two important examples of ϵ–relaxed logics are $\mathcal{SOLP}Horn[2]+\leq$ and $\mathcal{SOLP}Krom[2]+\leq$, which were defined and studied in [1] (see also section 2.1 above), and which capture **P** and **NL**, respectively. A sketchy proof of this assertion follows: For any $\epsilon < 1/2$, the problem NOT-IN-CLOS$_{\leq 1/2+\epsilon}$ (Example 1) is expressible in $(\mathcal{SOLP}Horn[2]+\leq)_\epsilon$, and it is complete for **P** via quantifier free first order reductions (same proof as in [1]). Therefore, any problem in **P** has a definition in $(\mathcal{SOLP}Horn[2]+\leq)_\epsilon$. Conversely, the satisfaction of sentences in $(\mathcal{SOLP}Horn[2]+\leq)_\epsilon$ can be decided in **P** by the algorithm described in [1] for $(\mathcal{SOLP}Horn[2]+\leq)$. Thus, $(\mathcal{SOLP}Horn[2]+\leq)_\epsilon = \mathbf{P} = \mathcal{SOLP}Horn[2]+\leq$. The argument for $(\mathcal{SOLP}Krom[2]+\leq)_\epsilon = \mathbf{NL} = \mathcal{SOLP}Krom[2]+\leq$ is similar.

The main property of relaxed fragments is the following:

Lemma 2. *Let $\mathcal{L}$ be a ϵ-relaxed fragment of $\mathcal{SOLP}$. Then for every sentence $\phi \in \mathcal{L}$, there exists a $\lambda \in (-\epsilon, \epsilon)$ and sentence $\theta \in \mathcal{L}_\lambda$ such that $\phi \leftrightarrow \theta$ and $\theta \Leftrightarrow_S \theta$.*

Proof. (See the Appendix.) □

The previous lemma motivates our notion of strong expressibility.

Definition 8. *Let $\mathcal{L} \subseteq \mathcal{L}' \subseteq \mathcal{SOLP}$ and fix $\phi \in \mathcal{SOLP}$ a sentence. We say that the fragment $\mathcal{L}$* **strongly expresses** *a sentence ϕ with respect to $\mathcal{L}'$ iff there exists a formula $\psi \in \mathcal{L}$ and a formula $\theta \in \mathcal{L}'$ such that $\theta \Leftrightarrow_S \psi$ and $\theta \leftrightarrow \phi$.*

Clearly, if a fragment $\mathcal{L}$ strongly expresses a sentence ϕ (with respect to any extension), then $\mathcal{L}$ expresses the sentence ϕ (because $\theta \Leftrightarrow_S \psi$ implies $\theta \leftrightarrow \psi$).

When we are working with relaxed fragments, we get the following strengthening of the above observations.

Theorem 2. *Let $\mathcal{L}, \mathcal{L}'$ be ϵ-relaxed fragments of $\mathcal{SOLP}$ such that $\mathcal{L} \subseteq \mathcal{L}'$ and let $\phi \in \mathcal{SOLP}$ a sentence. Then the following statements are equivalent:*

- *ϕ is expressible in $\mathcal{L}$;*
- *There exists a $\mu \in (-\epsilon, \epsilon)$ such that ϕ is strongly expressible in $\mathcal{L}_\mu$ with respect to $\mathcal{L}'_\mu$, i.e. there exists sentences $\rho \in \mathcal{L}', \theta \in \mathcal{L}_\mu$ such that $\phi \leftrightarrow \rho_\mu$ and $\rho_\mu \Leftrightarrow_S \theta$.*

Proof. (See the Appendix.) □

The importance of this theorem is that it shows the equivalence of the notion of expressibility and strong expressibility in the context of ϵ-relaxed fragments. This suggest that any tool that helps us prove strong inexpressibility may be transformed into a tool that proves inexpressibility. We work towards obtaining such a tool next.

Theorem 3. *Fix two ϵ-relaxed fragments $(\mathcal{L}+\leq) \subseteq (\mathcal{L}'+\leq)$ of $\mathcal{SOLP}+\leq$ and a sentence $\phi \in (\mathcal{L}'+\leq)$. Assume that for every formula $\theta \in (\mathcal{L}+\leq)$ and every $0 < \omega < \epsilon$ there exists a sublinear function g and two models $\mathcal{A}, \mathcal{B}$ in $(\mathcal{L}+\leq_g)$ (i.e. almost ordered models), such that:*

- *If $\mathcal{A} \models \theta$ then $\mathcal{B} \models \theta$;*
- *$\mathcal{A}/_{\sim_g} \models \phi$ and $\mathcal{B}/_{\sim_g} \not\models \phi$;*
- *if $|\mathcal{A}| = m_1$ and $|\mathcal{B}| = m_2$ then $g(m_i)/(2m_i - g(m_i)) < \omega$, for $i = 1, 2$.*

Then ϕ is not strongly expressible by $\mathcal{L}+\leq$.

Proof. (See the Appendix.) □

Observe that this theorem gives strong inexpressibility over ordered structures.

We will now use the previous theorem and Theorem 2 to obtain a tool, based on almost orders and ϵ–relaxed fragments, that gives inexpressibility of formulas over structures with built-in order; thus, providing a framework where proofs of non expressibility might be easier by doing some of the work in almost ordered structures

Theorem 4. *Fix two ϵ-relaxed fragments $(\mathcal{L}+\leq) \subseteq (\mathcal{L}'+\leq)$ of $(\mathcal{SOLP}+\leq)$ and a sentence $\phi \in (\mathcal{L}'+\leq)$. Assume that for every $\mu \in (-\epsilon, \epsilon)$, for every $\omega > 0$ such that $(\mu - \omega, \mu + \omega) \subseteq (-\epsilon, \epsilon)$, for every formula $\theta \in (\mathcal{L}_\mu + \leq)$ there exists a sublinear function g and two models $\mathcal{A}, \mathcal{B}$ in $(\mathcal{L}+\leq_g)$ such that:*

- *If $\mathcal{A} \models \theta$ then $\mathcal{B} \models \theta$;*
- *$\mathcal{A}/_{\sim_g} \models \phi$ and $\mathcal{B}/_{\sim_g} \not\models \phi$;*
- *if $|\mathcal{A}| = m_1$ and $|\mathcal{B}| = m_2$ then $g(m_i)/(2m_i - g(m_i)) < \omega$, for $i = 1, 2$.*

Then ϕ is not expressible in $(\mathcal{L}+\leq)$.

Proof. (See the Appendix.) $\square$

References

1. Arratia, A., and Ortiz, C. (2006) Expressive Power and Complexity of a Logic with Quantifiers that Count Proportions of Sets, *J. of Logic and Computation*, 16 (6): 817-840 .
2. Etessami, K., and Immerman, N. (1995), Reachability and the power of local ordering, *Theoretical Comp. Sci.* **148** (2): 261–279.
3. Grädel, E. (1992), Capturing complexity classes by fragments of second order logic, *Theoretical Comp. Sci.* **101**, 35–57.
4. Immerman, N., *Descriptive Complexity* (Springer, 1998).
5. Keisler, H.J., Hyperfinite model theory. In: R.C. Gandy and J.M. E. Hyland (eds) Logic Colloquium 76 (North-Holland,1977).
6. Libkin, L., and Wong, L., (2002) Lower bounds for invariant queries in logics with counting. *Theoretical Comp. Sci.* **288**, 153-180.

Technical Appendix

Proof of Theorem 1: By induction in the syntactic complexity of the formula.

First order quantification The key is that consistency with respect to $\leq_g$ holds for all formulas true in $\mathcal{A}$, and therefore, it is indistinct which representative of a $\sim_g$-class we take as witnesses for the existentially or universally quantified variables, together with the fact that, for any ϵ, the ϵ–approximation coincides with the original formula.

Proportional quantifiers (i): Suppose that $\mathcal{A}$ satisfies the formula $(P(Y) \geq r)\theta(\overline{a},\ \overline{S},\ Y)$ for $0 < r < 1$ and Y of arity $k \geq 1$. Then, for some $B \subseteq A^k$, $|B| \geq rm^k$ and $\mathcal{A} \models \theta(\overline{a}, \overline{S}, B)$. By inductive hypothesis $\mathcal{A}/_{\sim_g} \models \theta([\overline{a}]_g, [\overline{S}]_g, [B]_g)_{\gamma(m)}$, where $\gamma(m) = g(m)/(2m - g(m))$. In the worst case, B contains elements from every $\leq_g$–2-preorder, and when passing to its $\leq_g$-contraction, all possible equivalent k–tuples determined by elements in the same class are removed. There are at most $k(g(m)/2)m^{k-1}$ of these and thus, $|[B]_g| \geq rm^k - k\dfrac{g(m)}{2}m^{k-1}$ provided $rm > k(g(m)/2)$; otherwise, we can only say $|[B]_g| \geq 0$. The proportion of this set of $\sim_g$–classes with respect to the totality of k–tuples in $[A]_g$ is

$$
\begin{aligned}
P([B]_g) &\geq \left(\frac{2m}{2m - g(m)}\right)^{k-1} \left[r\left(\frac{2m}{2m - g(m)}\right) - k\frac{g(m)}{2m - g(m)}\right] \\
&= (1 + \gamma(m))^{k-1}[r(1 + \gamma(m)) - k\gamma(m)] \\
&= (1 + \gamma(m))^{k-1}[r - (k - r)\gamma(m)] \\
&\geq (1 - \gamma(m))^{k-1}[r - k\gamma(m)]
\end{aligned}
$$

Thus,

$$\mathcal{A}/_{\sim_g} \models (P(Y) \geq (1 - \gamma(m))^{k-1}[r - k\gamma(m)])\theta([\overline{a}]_g, [\overline{S}]_g, Y)_{\gamma(m)}$$

which is the desired result.

Now, suppose that $\mathcal{A}$ satisfies the formula $(P(Y) \leq r)\theta(\overline{a}, \overline{S}, Y)$, with r and Y as above. We argue similarly as before, but now the witness set B is such that, in the worst case, $|[B]_g| \leq rm^k$, which is the following proportion of $|[A]_g|^k = (m - g(m)/2)^k$:

$$
\begin{aligned}
P([B]_g) &\leq \left(\frac{2m}{2m - g(m)}\right)^{k-1} \left[r\left(\frac{2m}{2m - g(m)}\right)\right] \\
&= (1 + \gamma(m))^{k-1} r(1 + \gamma(m)) \leq (1 + \gamma(m))^{k-1}[r + k\gamma(m)]
\end{aligned}
$$

Thus,

$$\mathcal{A}/_{\sim_g} \models \left(P(Y) \leq (1 + \gamma(m))^{k-1}[r + k\gamma(m)]\right) \theta([\overline{a}]_g, [\overline{S}]_g, Y)_{\gamma(m)}$$

(ii): *(We omit due to space restrictions, but the strategy is similar to* (i)*.)*

(iii) and (iv): Follow from parts (i) and (ii) and that $(\theta_{-\epsilon})_\epsilon = \theta$. $\square$

Proof of Lemma 2: Fix a sentence $\phi \in \mathcal{L}$. For every $\delta \in (-\epsilon, \epsilon)$ there exists then a sentence $\phi[\delta] \in \mathcal{L}$ such that $(\phi[\delta])_\delta \leftrightarrow \phi$. The cardinality of all the sentences in $\mathcal{L}$ is countable. Hence by the pigeonhole principle there exists a sentence $\theta \in \mathcal{L}$ and two real numbers $\gamma < \delta \in (-\epsilon, \epsilon)$ such that $\theta_\gamma \leftrightarrow \phi \leftrightarrow \theta_\delta$. The Technical Lemma below shows that there exists λ, μ such that $\lambda \in (\gamma, \delta) \subseteq (-\epsilon, \epsilon)$, $\mu > 0$, and $\phi \to \theta_\gamma \to (\theta_\lambda)_{-\mu} \to \theta_\lambda \to (\theta_\lambda)_\mu \to \theta_\delta \to \phi$. Hence $\theta_\lambda \Leftrightarrow_S \theta_\lambda \leftrightarrow \phi$. □

Technical Lemma: For every formula $\theta(\overline{x}, \overline{X}) \in \mathcal{SOLP}(\tau)$, for every γ and λ, with $-1 \leq \gamma < \lambda \leq 1$, for every $\delta \in (\gamma, \lambda)$, there exists a $\mu > 0$ such that:

> $(\delta - \mu, \delta + \mu) \subseteq (\gamma, \lambda)$, and for every τ–structure $\mathcal{A}$ and every interpretation $\overline{A}$ of relation symbols $\overline{X}$ in $\mathcal{A}$, and elements $\overline{a}$ in $\mathcal{A}$, we have
>
> $$\mathcal{A} \models \theta(\overline{a}, \overline{A})_\gamma \to (\ \theta(\overline{a}, \overline{A})_\delta\ )_{-\mu} \to \theta(\overline{a}, \overline{A})_\delta \to (\ \theta(\overline{a}, \overline{A}\ )_\delta)_\mu \to \theta(\overline{a}, \overline{A})_\lambda$$

Proof: The proof is by induction in formulas.

Assume that the desired property holds for $\theta(\overline{x}, \overline{X}, Y)$, and consider the formula $\Psi(\overline{x}, \overline{X}) := (P(Y) \geq r)\theta(\overline{x}, \overline{X}, Y)$. Let

$$f(r, \omega) := \begin{cases} 1 & \text{if } \frac{r}{(1+\omega)^{k-1}} - k\omega \geq 1 \text{ and } \omega < 0 \\ \frac{r}{(1+\omega)^{k-1}} - k\omega & \text{if } \frac{r}{(1+\omega)^{k-1}} - k\omega \leq 1 \text{ and } \omega < 0 \\ (1-\omega)^{k-1}[r - k\omega] & \text{if } 0 \leq (1-\omega)^{k-1}[r - k\omega] \text{ and } \omega \geq 0 \\ \\ 0 & \text{if } (1-\omega)^{k-1}[r - k\omega] < 0 \text{ and } \omega \geq 0 \end{cases}$$

be a function from $[0, 1] \times (-1, 1)$ onto $[0, 1]$. Note that this function is continuous and for every $r \in [0, 1]$ and $\epsilon \in (-1, 1)$,

$$(\ (P(Y) \geq r)\theta(\overline{x}, \overline{X}, Y)\ )_\epsilon := (P(Y) \geq f(r, \epsilon)\ )(\theta(\overline{x}, \overline{X}, Y))_\epsilon.$$

Furthermore, for every $r \in [0, 1]$, $f(r, \ )$ is a decreasing function with the property that $f(r, 0) = r$. Fix then a nonempty interval $(\gamma, \lambda) \subseteq (-1, 1)$ and a $\delta \in (\gamma, \lambda)$. By induction hypothesis there exists a μ_1 with $(\delta - \mu_1, \delta + \mu_1) \subseteq (\gamma, \lambda)$ and such that for every model $\mathcal{A}$ and for every interpretations $\overline{A}$ and B of relation symbols and elements $\overline{a}$ in $\mathcal{A}$, we have that:

$$\mathcal{A} \models \theta(\overline{a}, \overline{A}, B)_\gamma \to (\ \theta(\overline{a}, \overline{A}, B)_\delta\ )_{-\mu_1} \to \theta(\overline{a}, \overline{A}, B)_\delta \to (\ \theta(\overline{a}, \overline{A}, B)_\delta\ )_{\mu_1} \to \theta(\overline{a}, \overline{A}, B)_\lambda.$$

Note that $f(f(r, \delta), 0) = f(r, \delta)$. Note also that $f(r, \lambda) \leq f(r, \gamma)$. Then, since f is continuous, there exists a μ_2 such that, for all $\epsilon \in [-\mu_2, \mu_2]$,

$$f(\ f(r, \delta), \epsilon) \in [f(r, \lambda), f(r, \gamma)]$$

Let $\mu = \min\{\mu_1, \mu_2\}$. From the previous remarks we know that $(\delta - \mu, \delta + \mu) \subseteq (\gamma, \lambda)$ and that for every model $\mathcal{A}$ and for every interpretations $\overline{A}$ of relation symbols and elements $\overline{a}$ in $\mathcal{A}$, we have that:

$$\begin{aligned} \mathcal{A} \models (P(Y) \geq f(r, \gamma))[\theta(\overline{a}, \overline{A}, Y)]_\gamma &\to (P(Y) \geq f(f(r, \delta), -\mu))[\theta(\overline{a}, \overline{A}, Y)_\delta\ ]_{-\mu} \\ \to (P(Y) \geq f(f(r, \delta), 0))[\theta(\overline{a}, \overline{A}, Y)]_\delta &\to\ (P(Y) \geq f(f(r, \delta), \mu))[\theta(\overline{a}, \overline{A}, Y)_\delta\ ]_\mu \\ &\to (P(Y) \geq f(r, \lambda))[\theta(\overline{a}, \overline{A}, Y)]_\lambda. \end{aligned}$$

but this is exactly

$$\mathcal{A} \models \Psi(\overline{a}, \overline{A})_\gamma \rightarrow (\ \Psi(\overline{a}, \overline{A})_\delta\ )_{-\mu} \rightarrow \Psi(\overline{a}, \overline{A})_\delta \rightarrow (\ \Psi(\overline{a}, \overline{A}\ )_\delta)_\mu \rightarrow \Psi(\overline{a}, \overline{A})_\lambda$$

(The case of formula $\Psi(\overline{x}, \overline{X}) := (P(Y) \leq r)\theta(\overline{x}, \overline{X}, Y)$ is treated similarly.) $\square$

Proof of Theorem 2: We show the hard direction. Assume that ϕ is expressible in $\mathcal{L}$. Note first that from lemma 2, since $\mathcal{L}'$ is an ϵ-relaxed fragment, we know that there exists a sentence $\rho \in \mathcal{L}'_\lambda$ for some $\lambda \in (-\epsilon, \epsilon)$ such that $\phi \leftrightarrow \rho$ and $\rho \Leftrightarrow_S \rho$. More specifically there exists γ such that $(\lambda - \gamma, \lambda + \gamma) \subseteq (-\epsilon, \epsilon)$ and $\rho_\gamma \leftrightarrow \rho_{-\gamma}$.

From hypothesis we know that there exists $\theta[\lambda] \in \mathcal{L}_\lambda$ such that $\theta[\lambda] \leftrightarrow \phi \leftrightarrow \rho$. Applying again lemma 2 to $\theta[\lambda]$ and using the fact that $\mathcal{L}_\lambda$ is a γ-relaxed fragment, we know that there exists a sentence $\theta \in \mathcal{L}_\mu$ for some μ in $(\lambda - \gamma, \lambda + \gamma)$ such that $\theta[\lambda] \leftrightarrow \theta$ and $\theta \Leftrightarrow_S \theta$. More specifically there exists ω such that $(\mu - \omega, \mu + \omega) \subseteq (\lambda - \gamma, \lambda + \gamma) \subseteq (-\epsilon, \epsilon)$ and $\theta_\omega \leftrightarrow \theta_{-\omega}$. We have then the following sequences of implications:

$$\rho_\gamma \rightarrow \rho_{-\gamma} \rightarrow \rho \rightarrow \theta[\lambda] \rightarrow \theta \rightarrow \theta_\omega \rightarrow \theta_{-\omega}$$

and symmetrically,

$$\theta_\omega \rightarrow \theta_{-\omega} \rightarrow \theta \rightarrow \theta[\lambda] \rightarrow \rho \rightarrow \rho_\gamma \rightarrow \rho_{-\gamma}$$

These two sequences of implications imply that $\rho_\mu \Leftrightarrow_S \theta$, with $\rho_\mu \in \mathcal{L}'_\mu$, $\theta \in \mathcal{L}_\mu$ and $\phi \leftrightarrow \rho_\mu$. $\square$

Proof of Theorem 3: Assume in order to get a contradiction that ϕ is strongly expressed in $(\mathcal{L}+ \leq)$. Then there exists sentences $\rho \in (\mathcal{L}'+ \leq)$, $\theta \in (\mathcal{L}+ \leq)$ with $\rho \leftrightarrow \phi$ and $\theta \Leftrightarrow_S \rho$.

We know then that there exists an $\omega \in (0, \epsilon)$ such that for every model $\mathcal{C}$ in $\mathcal{SOLP}+ \leq$ the following property $(*)$ holds:

- $\mathcal{C} \models \theta_\omega \rightarrow \rho_{-\omega}$,
- $\mathcal{C} \models \rho_\omega \rightarrow \theta_{-\omega}$.

Consider then the two models $\mathcal{A}, \mathcal{B}$ and the sublinear function g associated with $\phi, \epsilon, \theta, \omega$ by the hypothesis of the theorem. We consider two cases.

- If $\mathcal{A} \models \theta$ then by hypothesis we have that $\mathcal{B} \models \theta$. Applying now the Bridge Theorem we get that $(\mathcal{B}/\sim_g) \models \theta_\omega$. However, since $(\mathcal{B}/\sim_g) \not\models \phi$ and $\phi \leftrightarrow \rho$, we get that $(\mathcal{B}/\sim_g) \not\models \rho_{-\omega}$, but this contradicts property $(*)$.
- If $\mathcal{A} \not\models \theta$ then by the Bridge Theorem we have that $(\mathcal{A}/\sim_g) \not\models \theta_{-\omega}$. But by hypothesis $(\mathcal{A}/\sim_g) \models \phi$ and $\phi \leftrightarrow \rho$, hence we get that $(\mathcal{A}/\sim_g) \models \rho_\omega$, which is a contradiction with property $(*)$. $\square$

Proof of Theorem 4: Assume in order to get a contradiction that ϕ is expressible in $(\mathcal{L}+ \leq)$. Since $(\mathcal{L}+ \leq)\ \subseteq\ (\mathcal{L}'+ \leq)$ and $(\mathcal{L}+ \leq)$ and $(\mathcal{L}'+ \leq)$ are ϵ-relaxed we can invoke Theorem 2 to obtain that there exists a $\mu \in (-\epsilon, \epsilon)$ such that ϕ is strongly expressible in $(\mathcal{L}_\mu+ \leq)$ with respect to $(\mathcal{L}'_\mu+ \leq)$, i.e. there exists sentences $\rho \in (\mathcal{L}'+ \leq), \theta \in (\mathcal{L}_\mu+ \leq)$ such that $\phi \leftrightarrow \rho_\mu$ and $\rho_\mu \Leftrightarrow_S \theta$.

We know then that there exists a $0 < \omega < 1$ such that for every model $\mathcal{C}$ of $(\mathcal{SOLP}+ \leq)$: $\mathcal{C} \models \theta_\omega \rightarrow (\rho_\mu)_{-\omega}$ and $\mathcal{C} \models (\rho_\mu)_\omega \rightarrow \theta_{-\omega}$.

Note that we can select ω such that $(\mu - \omega, \mu + \omega) \subseteq (-\epsilon, \epsilon)$.

Consider then the two models $\mathcal{A}, \mathcal{B}$ and the sublinear function g associated to $\phi, \epsilon, \mu, \theta$ by the hypothesis of the theorem. We consider two cases.

- If $\mathcal{A} \models \theta$ then by hypothesis we have that $\mathcal{B} \models \theta$. Applying now the Bridge Theorem we get that $(\mathcal{B}/_{\sim_g}) \models \theta_\omega$. However, since $(\mathcal{B}/_{\sim_g}) \not\models \phi$ and $\phi \leftrightarrow \rho_\mu$, we get that $(\mathcal{B}/_{\sim_g}) \not\models \rho_\mu$, but this contradicts the hypothesis that $\rho_\mu \Leftrightarrow_S \theta$.
- If $\mathcal{A} \not\models \theta$ then by the Bridge Theorem we have that $(\mathcal{A}/_{\sim_g}) \not\models \theta_{-\omega}$. But by hypothesis $(\mathcal{A}/_{\sim_g}) \models \phi$ and $\phi \leftrightarrow \rho_\mu$, hence we get that $(\mathcal{A}/_{\sim_g}) \models (\rho_\mu)_\omega$, which is a contradiction with the hypothesis that $\psi \Leftrightarrow_S \theta$. $\square$